PROJECTIVE GEOMETRY: THE GEOMETRY BEHIND ALGEBRAIC EQUATIONS

Welcome to the Clubles de Ciencias en México! This club aims at exploring and unveiling the geometry behind polynomial equations. For example, 16th-century painters in Europe were concerned with projecting faithfully a 3D object onto a 2D plane. The mathematical model needed in order to carry this out came about 200 years later, and it is the so-called now Projective Geometry which is modeled using algebraic equations.

By combining modern mathematical tools, and the computer software Macaulay2, we plan to examine mathematical objects in dimension two, three, and if time permits, four.

This club will strongly encourage student participation and will be based upon mathematical collaboration and exploration.

1. First Day

What is it that this subject accomplishes?
Why do we examine these mathematical objects?

Activity (50min) Projecting a circle, and other curves, in $\mathbb{R}^3$ into a piece of paper. Drawing using rule and compass. How do you describe mathematically curves in $\mathbb{R}^3$?

Goal: think about the properties of the projected objects (can one transform one into another?).

Lecture (40min) Solving algebraic equations: complex numbers. Ring of polynomials and ideals. First case, linear equations: matrices and determinants.

Exploration (60min) Projecting a line and a conic in $\mathbb{R}^3$ into the xy-plane ($\cong \mathbb{R}^2$). Use these ideas in order to project to the xyz-plane the line defined by the following equations.

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= 0, \\ a_2x + b_2y + c_2z + d_2w &= 0, \\ a_3x + b_3y + c_3z + d_3w &= 0. \end{aligned} \tag{1}$$

Introduction to Macaulay2.

Discussion (30min) What is the geometric situation needed in order to project from a point and object in $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$? how do we guarantee that such a situation holds?

Conclusions (40min) : Lines project to lines. The situations needed is the uniqueness of the solution for a system of linear of n equations in n variables.

How do we think about projecting curves in $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$?

Film: David Eisenbud: Fundamental Theorem of Algebra (Numberphile)

Tomorrow : using matrices project a smooth conic to the form $x^2 + y^2 = 1$.

Lecture notes. The ring of polynomials with complex coefficients are denoted by $\mathbb{C}[x_0, \dots, x_n] = R$. It is a called a ring because we can add and multiply elements. In it, we have ideals $\mathcal{I} \subset R$. An ideal $\mathcal{I}$ is closed under addition, and $g \in \mathcal{I}$, $f \in R$ implies $f \cdot g \in \mathcal{I}$.

Definition 1.1. An ideal $I \subset R$ will be called prime, if $fg \in I$ implies that either $f \in I$ or $g \in I$.

2. Second Day

Activity (50min)	Describe three things we did yesterday. Proof the Fundamental Theorem of Algebra.
Lecture (40min)	Projective plane. Projective transformations (lines map to lines and conics to conics) and composition (matrix multiplication). Conics and symmetric matrices. Smoothness of a conic.
Exploration (60min)	Let $C = \{x^2 + y^2 + xy + x + y = c\}$. Eliminate xy. Then, proceed to eliminate x, y. What happens to the matrix associated to C? Can we obtain a diagonal matrix always? What does this say about the projective properties of conics? What happens to the derivatives?
Discussion (30min)	All the conics are the same projectively. What does this mean (analyze the geometry)? What happens to the associated symmetric matrix?
Conclusions (40min)	:

How do we think about distinct conics in the plane?

Film:	David Eisenbud talks about equations of degree 3.
Tomorrow	Hilbert polynomial and the genus of an algebraic curve.

Lecture notes. Let us define the projective space. Let $V = \mathbb{C}^{n+1}$ be the vector space of dimension $n + 1$. Consider the set of all subspaces of dimension 1 in V. Such a set is denoted by $\mathbb{P}^n$, and is called projective space. It has global homogeneous coordinates,

$$\mathbb{P}^2 = \{[x : y : z] | [x : y : z] = [ax : ay : az]\},$$

for a nonzero scalar $a \in \mathbb{C}$.

Let ϕ be a ring homomorphisms $\phi : R \to R$. Then $\phi(fg) = \phi(f)\phi(g)$, and $\phi(f + g) = \phi(f) + \phi(g)$. Linear are homomorphisms.

3. Third Day

Lecture (40min)	Graded modules and Hilbert polynomial. Arithmetic genus of a curve.
Exploration (60min)	Macaulay2: List the arithmetic genus possible for plane algebraic curves. Can we define a plane algebraic curve using two equations on $\mathbb{P}^2$?
Lecture (30min)	Graded modules and Hilbert polynomial. Arithmetic genus of a curve. Transversality and irreducibility of an algebraic set.
Discussion (60min)	Find the Hilbert polynomial for a curve in space. Explain what's goes on at the level of ideals an modules. Dimension of an algebraic variety. Guess a formula for the genus of a plane curve.
Conclusions (40min)	:

How do we think about the genus of a curve?

Film:	Joe Harris talks about the arithmetic genus.
Tomorrow	Curves in space and their invariants.

Lecture notes. The ring of polynomial $R = \mathbb{C}[x, y, z] = \bigoplus R_k$ has a natural grading in terms of the degree. Let $\mathcal{I} \subset \mathbb{C}[x, y, z]$ be an ideal. Then

$$M = \mathbb{C}[x, y, z]/\mathcal{I} = \bigoplus M_k$$

is a graded $\mathbb{C}[x, y, z]$-module. The Hilbert polynomial is defined by

$$HP(n) = \dim M_n$$

For example, the ideal

$$\mathcal{I} = (x^2 + y^2 + z^2) \subset \mathbb{C}[x, y, z]$$

has the Hilbert polynomial: $HP_{\mathcal{I}}(n) = 2n + 1$.

Definition 3.1. Let $X \subset \mathbb{P}^n$ be a projective variety. The **dimension** of X is defined as the degree of the Hilbert polynomial $HP_{\mathcal{I}}(n)$, where $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ stands for the ideal of X.

Definition 3.2. A projective algebraic variety of dimension 1 will be called **curve**. Similarly, a projective algebraic variety of dimension 2 is called a **surface**.

Definition 3.3. Let $X \subset \mathbb{P}^n$ be a projective curve. The **arithmetic genus** X is defined to be the number $1 - HP_{\mathcal{I}}(0)$, where $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ stands for the ideal of X.

Definition 3.4. The variety $X \subset \mathbb{P}^3$ is irreducible, if its ideal $\mathcal{I}_X \subset \mathbb{C}[x, y, z, w]$ is prime.

Example: Consider the variety in $\mathbb{P}^3$ defined by the ideal $X = (xz - w^2, xz - yw)$. This variety consists of a conic and two lines.

4. Fourth Day

Explorations (40min)	Macaulay2: Projecting curves from $\mathbb{P}^n$ to $\mathbb{P}^3$ and $\mathbb{P}^2$. Smoothness.
Lecture (40min)	Linear projections. Secants and multi-secants. Smoothness of a curve. Singular double points on plane curves.
Exploration (30min)	Project a smooth curve in $\mathbb{P}^3$ of degree d and genus g to $\mathbb{P}^2$. can you guess what is the relation among g, d and $\delta =$(double points)?
Discussion (40min)	Curve of genus two in $\mathbb{P}^3$: Project it to $\mathbb{P}^2$ and get a plane quintic. Is this a contradiction to the formulas above? Origin and geometry of singularities on the projected curves. Is there a maximal number of nodes that a plane curve can have?
Conclusions (40min)	:

How do we think about the singularities of plane curves?

Film:	Joe Harris talks about curves in $\mathbb{P}^3$.
Tomorrow	Implications of what we've done thus far.

Lecture notes. All the functions in $\mathbb{A}^2$ whose 1st-derivative (and not higher order) vanishes at $(0,0)$ have generators [Hulek, pag. 202],

$$\mathfrak{m}^2_{\mathbb{P}^2,p}/\mathfrak{m}^3_{\mathbb{P}^2,p} = (x^2, xy, y^2).$$

A function f in this ring vanishes at zero, and so do its derivatives.

Let $C = \{f = 0\}$ be a plane curve. We say C is smooth at $p \in C$ if $\frac{\partial f}{\partial x_i}(p) \neq 0$ for some i.

Definition 4.1. Let $p \in C \subset \mathbb{P}^2$ be a point on a plane curve C. The the local ring of C at p is defined to be $\mathfrak{m}_{C,p} := \{g \in \mathbb{C}[C] \mid g(p) = 0\}$.

Definition 4.2. The curve $C = \{f = 0\}$ is said to have a node at the point $p \in C$ if $f \in \mathfrak{m}^2_p/\mathfrak{m}^3_p$, and if

$$\overline{f} \in \mathfrak{m}^2_p/\mathfrak{m}^3_p$$

has two distinct components.

Example: Let $C = \{y^2 = x^3 + x^2\} \subset \mathbb{A}^2$ be an affine plane curve. Since $\partial_x C = 3x^2 + 2x$, and $\partial_y C = 2y$, then $f \in \mathfrak{m}^2_{C,(0,0)}$. Furthermore, $\overline{f} \in \mathfrak{m}^2_p/\mathfrak{m}^3_p$ is equal to $y^2 - x^2 = (y-x)(x+y)$, which consists of two distinct lines. Hence, C has a node at $(0,0)$.

References

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[Reid] M. Reid, *Undergraduate Algebraic Geometry*, London Mathematical Society, 1998.
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